Quantum Imaging using Non-linear Optics

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1 Introduction and Motivation

Imaging using entangled photons is a very interesting phenomenon, [1], and it throws light into the fundamental aspects of quantum mechanics itself. Spontaneous parametric down conversion (SPDC) is a powerful tool for generating these entangled states. In SPDC, an intense laser beam, called the pump, is incident upon an anisotropic crystal. The pump is intense enough so that the nonlinear effects lead to conversion of a pump photon into a pair of correlated or entangled photons also called biphotons. The reason for this entanglement in spatial and spectral domains is the conservation of energy and momentum for each photon pair. The entangled photons can be generated in two possible configurations, type I and type II. In type I phase matching, both the signal and idler photons have parallel polarizations and in type II configuration, the signal and idler photons have perpendicular polarizations. In either case, the following phase matching conditions must be valid:

$$\omega_p = \omega_s + \omega_i, \tag{1}$$

$$\mathbf{k}_p = \mathbf{k}_s + \mathbf{k}_i, \tag{2}$$

where subscripts p, s and i stand for pump, signal and idler respectively. These are the enegery and momentum conservations of the photons concerned and the basis for entanglement of the signal and idler photons. We will see how there entangled photons provide a way to image an object without seeing it.

For example, as illustrated in Fig. 1, consider a 3-D object placed within a chamber that has an opening through which light enters but does not escape, [2]. The wall of the chamber is coated with a photosensitive surface and serves as an integrating sphere that converts any photon reaching it into a photoevent. The chamber therefore serves as a photon bucket that indiscriminately detects the arrival of photons at any point on its surface, whether scattered or not, but is totally incapable of discerning the location at which the photon arrives. Classically, whatever the nature of the light source or the construction of the imaging system, it is impossible to construct a hologram of the 3-D object in this configuration. This is because optical systems that make use of classical light sources, even those that involve scanning and time-resolved imaging, are incapable of resolving the ambiguity of positions from which the photons are

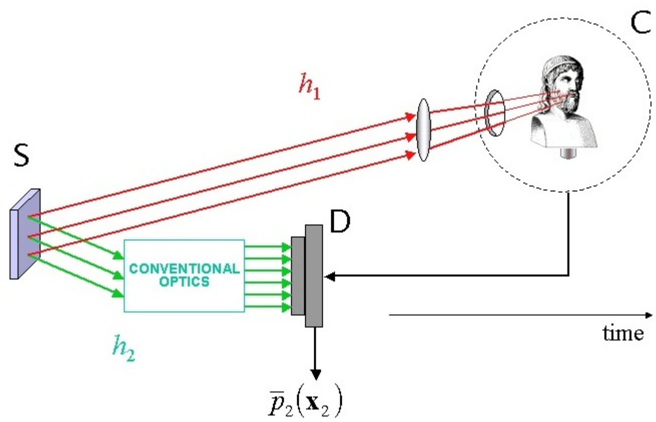

Figure 1: Figure showing the working of Quantum holography. S is a source of entangled-photon pairs, for example a nonlinear crystal. C is a (remote) single-photon-sensitive integrating sphere that comprises the wall of the chamber concealing the hidden object (bust of Plato). D is a (local) 2-D single-photon-sensitive scanning or array detector. h_1 and h_2 represent the optical systems that deliver the entangled photons from S to C and D, respectively. The quantity, $p_2(x_2)$ is the marginal coincidence rate given by Eq. (24), which is the hologram of the concealed object. Figure taken from [2].

scattered; they therefore cannot be used to form a coherent image suitable for holographic reconstruction. The implementation of quantum holography makes use of entangled-photon beams generated. As shown in Fig. 1, one beam from the source S enters the chamber opening and is scattered from the object, yielding a single sequence of photoevents from the integrating sphere C. The other beam is transmitted through a conventional optical system and detected using a single-photon-sensitive scanning (or array) detector D. The information registered by the two detectors, in the form of coincidence counts, is sufficient to extract coherent information about the 3-D object that is suitable for holographic reconstruction.

Even though most aspects of ghost imaging can be understood classically with thermal or pseudo-thermal light sources, ghost imaging performed with entangled photons can show intrinsically quantum features such as imaging in either near or far field, violating Bell inequalities. In this paper we will not debate whether there is a need for quantum entanglement to perform correlation imaging, but we will try to understand how the spontaneous parametric downconversion results in an entangled state and how this entangled state is able to perform ghost imaging. We will look into some correlation functions proposed by Glauber to understand the coincicence counts at the two detectors. Finally, we will see how we can apply these correlation functions to perform and understand various imaging system configurations.

2 Brief Theory of Spontaneous Parametric Down Conversion

To understand how SPDC generates entangled photons, we need to look into the quantum picture and quantized fields. Consider an electric field operator $\mathbf{E}(\mathbf{r}t)$ and its seperation into positive and negative frequency parts. The field operator has the following Fourier representation:

$$\mathbf{E}(\mathbf{r}t) = \int_{-\infty}^{\infty} \mathbf{e}(\omega, \mathbf{r}) \exp^{-i\omega t} d\omega, \tag{3}$$

where $\mathbf{e}(-\omega, \mathbf{r}) = \mathbf{e}^{\dagger}(\omega, \mathbf{r})$. Now, we can write the positive frequency part of $\mathbf{E}$ as

$$\mathbf{E}^{(+)}(\mathbf{r}t) = \int_{0}^{\infty} \mathbf{e}(\omega, \mathbf{r}) \exp^{-i\omega t} d\omega, \tag{4}$$

and the negative frequency part as

$$\mathbf{E}^{(-)}(\mathbf{r}t) = \int_{-\infty}^{0} \mathbf{e}(\omega, \mathbf{r}) \exp^{-i\omega t} d\omega, \tag{5}$$

$$= \int_{0}^{\infty} \mathbf{e}^{\dagger}(\omega, \mathbf{r}) \exp^{i\omega t} d\omega. \tag{6}$$

From the way the positive and negative frequency parts are defined we have $\mathbf{E} = \mathbf{E}^{(+)}(\mathbf{r}t) + \mathbf{E}^{(-)}(\mathbf{r}t)$, and $\mathbf{E}^{(-)}(\mathbf{r}t) = \mathbf{E}^{(+)\dagger}(\mathbf{r}t)$. We can show that the positive frequency part, $\mathbf{E}^{(+)}(\mathbf{r}t)$, is associated with photon absorption, that is, with the annihilation operator and the negative frequency part, $\mathbf{E}^{(-)}(\mathbf{r}t)$, is associated with photon emission, that is with the creation operator. With the help of these operators we can now write the Hamiltonian, [3], for the nonlinear interaction term, for type II configuration, as

$$H = \epsilon \int_{V} d^3r \chi E_p^{(+)} E_o^{(-)} E_e^{(-)} + H.c., \tag{7}$$

where V is the volume of the crystal illuminated by the pump laser E_p, E_o and E_e correspond to the ordinary ray and the extraordinary ray, respectively, χ is the nonlinear electric susceptibility tensor, and H.c. means the Hermitian conjugate. As we can see from Eq. (7), one pump photon is destroyed and two photons are created in this downcoversion process. The Hermitian conjugate term has the reverse process, that it, two photons will be destroyed and a sum frequency photon will be generated.

From first order perturbation theory, we can express the state in terms of the Hamiltonian as as

$$|\Psi\rangle = |0\rangle - \frac{i}{\hbar} \int dt\, H|0\rangle. \tag{8}$$

Using the Hamiltonian given by Eq. (7), the two photon state at the output surface of the crystal can be written as

$$|\Psi\rangle = |0,0\rangle + \sum_{k_o,k_e} F_{k_o,k_e} a^{\dagger}_{ok_o} a^{\dagger}_{ek_e} |0,0\rangle, \tag{9}$$

where F_{k_o,k_e} is proportional to pump amplitude, E_0, $\delta(\omega_{ok_o} + \omega_{ek_e} - \omega_p)$ and $h(k_o + k_e - k_p)$. Note that h is not a delta function because of the finite length of the crystal. As we can see from Eq. (9), the two photon state is entangled because it can not be written in term of a simple product of two individual photon states. The coupling happens in the δ funtion and the h function. The first term is just the vaccum state and can be ignored for coincidence count measurements. The form of the equation gives us an idea of the general entangled state and in the following section, we will qualitatively see why an entangled state offers an advantage in overcoming the limitations of uncertainity relations imposed on separable states.

3 Entangled versus separable systems

We know that the position, x, and momentum, p, form a conjugate pair and their variances are limited by the uncertainity relation:

$$\Delta x \Delta p \geq \hbar. \tag{10}$$

As we are interested in the relative position and the sum of the momentum for a system of two photons, entangled and separable, we will study the corresponding uncertainity relations for both entangled and separable systems. Motivated by Eq. (9), consider the following pure state for a two-particle system entangled in momentum variables (p_1, p_2), [4],

$$|\Psi\rangle = \int\int dp_1 \; dp_2 \; \tilde{A}(p_1 + p_2)\tilde{C}(\frac{p_1 - p_2}{2})|p_1, p_2\rangle, \tag{11}$$

with the assumption that standard deviation (σ_A) of $\tilde{A}$ is much smaller than the standard deciation (σ_C) of $\tilde{C}$. We can then write the probability amplitudes in position and momentum representations are given by:

$$P(x_1, x_2) = A(\frac{x_1 + x_2}{2})C(x_1 - x_2) \tag{12}$$

$$\tilde{P}(p_1, p_2) = \tilde{A}(p_1 + p_2)\tilde{C}(\frac{p_1 - p_2}{2}) \tag{13}$$

Note that $(p_1 + p_2, (x_1 + x_2)/2)$ and $((p_1 - p_2)/2, x_1 - x_2)$ are pairs of Fourier conjugate variables whereas $(x_1 - x_2)$ and $(p_1 + p_2)$ are not. As a result we have

$$\Delta(p_1 + p_2) = \sigma_A \ll \Delta p_{12}, \tag{14}$$

$$\Delta(x_1 - x_2) = \frac{\hbar}{\sigma_C} \ll \Delta x_{12}, \tag{15}$$

where $\Delta p_{12} = 1/2\sqrt{\sigma_A^2 + \sigma_C^2}$ and $\Delta x_{12} = \hbar/(2\sigma_A\sigma_C)\sqrt{\sigma_A^2 + \sigma_C^2}$ are the minimum uncertainities associated with each particle of the entangled system. Since $(x_1 - x_2)$ and $(p_1 + p_2)$ are not conjugate to each other, their variances can be indefinitely small even though the indicidual uncertainities can be large, and an increase or decrease of one does not affect the other. Therefore, entangled two-particle states may give rise to comepletely independent uncertainties for total momentum and relative position. On the other hand, for systems of independent quanta, we have the following relations;

$$\Delta(p_1 \pm p_2) = \sqrt{\Delta p_1^2 + \Delta p_2^2},$$
$$\Delta(x_1 \pm x_2) = \sqrt{\Delta x_1^2 + \Delta x_2^2}, \tag{16}$$

where Δp_j, Δx_j are the uncertainities in momentum and position with the particle j $(j = 1, 2)$. The uncertanity principle will therefore, result in the following relation for variances of $(x_1 - x_2)$ and $(p_1 + p_2)$

$$\Delta(x_1 - x_2)\Delta(p_1 + p_2) \geq \hbar. \tag{17}$$

Even though $(x_1 - x_2)$ and $(p_1 + p_2)$ are not Fourier-conjugate variables, for independent quanta, the inequality (17) must hold. This means any attempt to reduce the uncertainity in joint momentum of the two particles will automatically cause an icrease in error for the joint measurement in position.

Till now we looked qualitatively why the entangled state provides an advantage over the independent quanta in terms of uncertainity relations. To quantitatively study the correlations among the fields at the two detectors, we need to find the field correlations at the two detectors for the two photons. In this regard, we will look into $G^{(2)}$ function proposed by Glauber, [5], in the following section.

4 Field Correlations

The properties of system of photon pairs can be understood with the help of coincidence between two spatially separated single-photon detectors. Rate at which a detector records photons is proportional to $\langle i|E^{(-)}(\mathbf{r}t)E^{(+)}(\mathbf{r}t)|i\rangle$, when the initial state of the field is $|i\rangle$. This is the probability per unit time that a photon will be absorbed by an ideal detector at point $\mathbf{r}$ and time t. When we use two detectors situated at different points $\mathbf{r}$ and $\mathbf{r'}$ to detect photon coincidences or more generally, delayed coincidences, the probability for such a detection is given by $\langle i|E^{(-)}(\mathbf{r}t)E^{(-)}(\mathbf{r'}t')E^{(+)}(\mathbf{r'}t')E^{(+)}(\mathbf{r}t)|i\rangle$. When we dont have a complete knowledge of the state of the field, we look into the density operator, $\hat{\rho}$, which specifies the system in terms of probabilities. The expectation value of an observable, $\hat{O}$ whose state is not known, can be obtained by averaging over the probability of states. The resulting average can be written as $tr\{\hat{\rho}\hat{O}\}$. If the fields $E^{(-)}$ and $E^{(+)}$ are evaluated at different points then we get a measure for the correlations of the complex fields at seperated positions and times. Then the correlation function for field can be written as

$$G^{(1)}(\mathbf{r}t, \mathbf{r'}t') = tr\{\hat{\rho}E^{(-)}(\mathbf{r}t)E^{(+)}(\mathbf{r'}t')\}. \tag{18}$$

In a similar fashion, generalizing the photon coincidence rate, we get the second-order correlation function,

$$G^{(2)}(\mathbf{r}_1 t_1, \mathbf{r}_2 t_2, \mathbf{r}_3 t_3, \mathbf{r}_4 t_4) = \\ tr\{\hat{\rho} E^{(-)}(\mathbf{r}_1 t_1) E^{(-)}(\mathbf{r}_2 t_2) E^{(+)}(\mathbf{r}_3 t_3) E^{(+)}(\mathbf{r}_4 t_4)\}. \tag{19}$$

For our purposes, we just look into rate of coincidence counts at two detectors, so that Eq. (19) becomes

$$G^{(2)}(\mathbf{r}_1 t_1, \mathbf{r}_2 t_2) = \\ tr\{\hat{\rho} E^{(-)}(\mathbf{r}_1 t_1) E^{(-)}(\mathbf{r}_2 t_2) E^{(+)}(\mathbf{r}_2 t_2) E^{(+)}(\mathbf{r}_1 t_1)\}. \tag{20}$$

Since we have the entangled state wave function from Eq. (9), we can calculate the rate of coincidence from Eq. (20). We can also compute marginal coincidence rates and conditional coincidence rates which can be used to measure the resolution of a particular imaging system. We will now look into the general formalism to do such calculations.

5 Fourier Optics

Consider the situation depicted in Fig. 5, [6], where the pump beam illuminates the nonlinear crystal (NLC), and the signal and idler beams are measured by the single-photon detectors D_1, placed at x_1 and D_2 placed at x_2, respectively. We assume throughout a planar source and a one-dimensional geometry in the transverse plane for the sake of simplicity. The time dependence in the $G^(2)$ function can be dropped as a result and the coincidence rate of photon pairs at the two detectors D_1 and D_2 is given by

$$G^{(2)}(x_1, x_2) = |\psi(x_1, x_2)|^2, \tag{21}$$

where the biphoton amplitude $\psi(x_1, x_2)$ is

$$\psi(x_1, x_2) = \int dx\, E_p(x) h_s(x_1, x) h_i(x_2, x). \tag{22}$$

The two functions h_s and h_i are the impulse-response functions for the signal and idler beams and $E_p(x)$ is the spatial distribution of the pump field at the crystal entrance. The relation Eq. (22) can be obtained by substituting the pure entangled state given by Eq. (9) into the $G^{(2)}$ function given by Eq. (20). We will need the following additional relation which relates the positive frequency part of the field, $E^{(+)}(x)$ at the detector plance to the annihilation operator, a, at the source plane of the crystal in terms of the impulse response function:

$$E^{(+)}(x) = \int h(x, x') a(x') dx'. \tag{23}$$

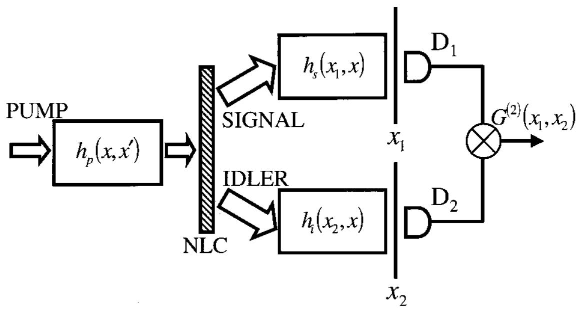

Figure 2: Figure showing the biphoton imaging, see text for details. Figure taken from [6].

Once we have the $G^{(2)}$ function, we can now define two other correlation functions, the marginal coincidence rate, $I^{(2)}(x_2)$, and the conditional coincidence rate, $I_0^{(2)}(x_2)$

$$I^{(2)}(x_2) = \int dx_1\, G^{(2)}(x_1, x_2), \tag{24}$$

$$I_0^{(2)}(x_2) = G^{(2)}(0, x_2). \tag{25}$$

The marginal rate, $I^{(2)}(x_2)$, gives us the rate at which D_2 detects a photon at x_2 when D_1 detects a photon at any location. The conditional rate, $I_0^{(2)}(x_2)$, gives us the rate of detecting at photon at x_2 by D_2 when a photon is recorded by D_1 at position, $x_1 = 0$. If we assume an object in the signal path, then by considering a delta function to be the object,we can measure the resolution of the imaging system.

With this formalism we can now measure resolution of biphoton imaging and also study various imaging configuration that are possible.

6 Summary

In this term paper, we looked into quantum imaging made possible by the entangled-photon pair generated by a SPDC process. We briefly looked into the theory of SPDC and then qualitatively understood the advantage of using an entangled pair over individual photons for imaging in terms of uncertainity relations. For quantitatively understanding the correlations between the fields, we looked into the coincident rates and then looked into the general framework to apply the coincidence rates for practical imaging situations.

7 References

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