Astronomy 421

Lecture 5: Radiation
Kirchhoff’s laws:

1. A hot, dense gas or hot solid object produces a continuous spectrum with no dark spectral lines.
2. A hot, diffuse gas produces bright spectral lines (emission lines).
3. A cool, diffuse gas in front of a source of a continuous spectrum produces dark spectral lines (absorption lines) in the continuous spectrum.

Empirical! What is the physical basis?

First law: This was the topic of the last lecture… Continuous spectrum of BB radiation $B_\lambda(T)$ or $B_\nu(T)$ emitted at any $T>0$ K.
Second and third laws awaited quantum mechanics for explanation:

Planck introduced idea that radiation was quantized to explain frequency distribution of radiation from a black body. He made an ad hoc assumption that radiation was in discrete quanta of size $h\nu$.

$$E = h\nu = \frac{hc}{\lambda}$$

…direct evidence for photon energy quantization had to wait for Einstein.
The photoelectric effect:

Bombard a metal with light of a given frequency and measure KE of any ejected electrons.

Red light did not cause any photoelectrons (regardless of intensity).

Violet light ejected a few photoelectrons.

The maximum electron KE increased when higher frequency photons used.

Thus: ejection energy independent of the total energy of illumination at a given frequency.

=> the interaction must be like that of a particle giving all of its energy to the electron.

The photoelectric effect can be understood only if light comes in discrete packets (photons) of energy => light is a particle.
Further evidence: change in $\lambda$ due to scattering off electron - the Compton effect. If photons are massless particles, not only is $E=\hbar \nu$, but then $E=\hbar c/\lambda = pc$ for particles at the speed of light (see Ch 4). If true, when photon scatters off electron at rest, photon energy and wavelength change, given by:

$$\Delta \lambda = \left( \frac{\hbar}{m_e c} \right) (1 - \cos \theta)$$

Compton experimentally verified this equation.

So photons have momentum $p = \hbar / \lambda$ even though massless.

*One astrophysical manifestation: radiation pressure.*
More on wave-particle duality:

de Broglie proposal (also true for massive particles):

they also exhibit wavelike behavior with a characteristic wavelength given by their momentum

\[ \lambda = \frac{h}{p} = \frac{h}{mv} \]

*de Broglie relation*

Confirmed in electron double-slit experiment.
**Wave-particle duality and H-atom energy levels:**
(alternative to C+O’s Bohr treatment to get energy levels).

Bohr accounted for the structure of H atoms by postulating that electron orbits are circular with quantized angular momentum and don’t radiate, except when they jump from high to low \( n \) levels.

De Broglie showed that this was equivalent to a standing wave condition, which looked ahead to wave functions and the Schrödinger Equation.

\[
\lambda = \frac{\hbar}{p}
\]

\( \lambda \) is the de Broglie wavelength

Diagram showing allowed and not allowed orbits.
de Broglie’s standing wave condition:

\[ 2\pi r = n\lambda \]

Since \( \lambda = \frac{h}{p} = \frac{h}{m_e v} \) \( \Rightarrow (2\pi)^2 r^2 = \frac{n^2 h^2}{m_e^2 v^2} \)

where \( m_e = e^- \) mass, \( v = \) orbital velocity.

Balancing centripetal force & Coulomb attraction:

\[ \frac{m_e v^2}{r} = \frac{e^2}{4\pi \epsilon_0 r^2} \Rightarrow v^2 = \frac{e^2}{4\pi \epsilon_0 m_e r} \]

Substituting for \( v^2 \):

\[ (2\pi)^2 r^2 = \frac{n^2 h^2}{m_e^2 e^2} \frac{4\pi \epsilon_0 m_e r}{e^2} \]
Then:

\[ r = \frac{4\pi\epsilon_0}{m_e e^2} \left( \frac{\hbar}{2\pi} \right)^2 n^2 = \frac{4\pi\epsilon_0 \hbar^2}{m_e e^2} n^2 = a_o \times n^2 \]

where \( n \) = principal quantum number and \( a_o \) is the Bohr radius.

(NB: C&O carefully do this with reduced mass, \( \mu = 0.9994556 \ m_e \),
in their Bohr treatment. Close enough.)
What is the energy of the $e$ "orbits"?

$$E = \frac{1}{2}m_e v^2 - \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r}$$

we know: \[ m_e v^2 = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r} \]

$$\Rightarrow E = \frac{1}{2} \left( \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r} \right) - \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r} = -\frac{1}{2} \left( \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r} \right)$$

since $r = 4\pi\varepsilon_0 n^2 \left( \frac{\hbar^2}{m_e e^2} \right)$

$$E = -\frac{m_e e^4}{32\pi^2 \varepsilon_0^2 \hbar^2} \frac{1}{n^2} = -13.6\text{eV} \frac{1}{n^2}$$

From balancing centripetal and Coulomb forces (or from Virial thm)

Quantized energy of an hydrogen atom (again should use reduced mass). Why negative?
So energy of a photon produced by a "jump".

\[ E_{\text{photon}} = E_{\text{high}} - E_{\text{low}} \]

leading to an expression for wavelengths:

\[
E_{\text{photon}} = \frac{hc}{\lambda} = \left( -\frac{m_e e^4}{32\pi^2 e^2 \hbar^2 n_{\text{high}}^2} \right) - \left( -\frac{m_e e^4}{32\pi^2 e^2 \hbar^2 n_{\text{low}}^2} \right)
\]

\[
\Rightarrow \frac{1}{\lambda} = \frac{m_e e^4}{64\pi^3 e^2 c \hbar^3} \left( \frac{1}{n_{\text{low}}^2} - \frac{1}{n_{\text{high}}^2} \right)
\]

or \[
\frac{1}{\lambda} = R_H \left( \frac{1}{n_{\text{low}}^2} - \frac{1}{n_{\text{high}}^2} \right)
\]

where \( R_H \) is the Rydberg constant for hydrogen.
H atom emission and absorption lines:

<table>
<thead>
<tr>
<th>Series</th>
<th>( n_{\text{low}} )</th>
<th>( n_{\text{high}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lyman</td>
<td>1</td>
<td>( \geq 2 ) (UV)</td>
</tr>
<tr>
<td>Balmer</td>
<td>2</td>
<td>( \geq 3 ) (Optical)</td>
</tr>
<tr>
<td>Paschen</td>
<td>3</td>
<td>( \geq 4 ) (IR)</td>
</tr>
<tr>
<td>Brackett</td>
<td>4</td>
<td>( \geq 5 ) (IR)</td>
</tr>
<tr>
<td>Pfund</td>
<td>5</td>
<td>( \geq 6 ) (IR)</td>
</tr>
</tbody>
</table>

Key lines:
- Ly\( \alpha \) (2->1) 121.6 nm
- H\( \alpha \) (3->2) 656.3 nm
- H\( \beta \) (4->2) 486.1 nm
Ionization and Recombination

Ionization: removing an electron by absorption of a photon or a collision. Photon, for example, must have enough energy to increase electron’s energy to zero. Any additional energy goes into KE of electron (remember photoelectric effect).

“Re”combination: electron rejoins atom or ion. Can join to any state (dictated by QM probabilities), but will cascade to ground state by various paths, emitting photons as it does so. Hα in HII regions comes from this, not excitation from lower levels, as we’ll see.
Heisenberg's uncertainty principle:

We cannot say with 100% certainty where a particle is and what its energy is.

\[ \Delta x \Delta p_x \geq \frac{\hbar}{2} \]

"Nature is intrinsically fuzzy"

Often you will see this form for making estimates:

\[ \Delta x \Delta p_x \approx \hbar \]

Or, in terms of energy and time:

\[ \Delta E \Delta t \approx \hbar \]

The last statement means that spectral lines cannot be perfectly sharp. This is called natural broadening.
Of course, we know that the Bohr model is not fully quantum mechanical. Orbits are more like fuzzy clouds of probability - the square of wave amplitude at a certain place describes the probability of finding a particle there.

Solution of the Schrödinger equation:
- same allowed energies as Bohr
- but also quantization of:
  a) orbital angular momentum

\[ L = \sqrt{l(l + 1)} \hbar \quad l = 0, 1, 2, \ldots n - 1 \]

b) \( z \) component of \( L \)

\[ L_z = m_l \hbar \quad m_l = 0, \pm 1, \pm 2, \ldots \pm l \]

c) spin angular momentum

- Fermions: \( m_s = \pm \frac{1}{2} \) e.g. e, p, n
- Bosons: \( m_s = \pm 1, 0 \) e.g. photons
Fermions obey the Pauli exclusion principle:

No two fermions can occupy the exact same quantum state (completely defined by $n, l, m_l, m_s$).

Explains properties of periodic table (electron shells). But also important for structure of dense stellar cores, white dwarfs, and neutron stars.
We will spend time on practical applications of quantum mechanics to stellar atmospheres, since the study of spectral lines yields info on:

- composition
- temperature
- density
- pressure
- bulk motion
- rotation
- magnetic fields

A similar bonanza awaits for study of emission lines from HII regions, planetary nebulae, supernova remnants, AGN - anything with spectral lines!
Example: Resolving power of a lens.

From the wave theory of light, the smallest angle a telescope can resolve is

\[ \theta_{\text{diff}} \approx 1.22 \frac{\lambda}{D} \]

where \( D \) is the diameter of the telescope, and \( \lambda \) is the wavelength of the EM radiation.

We can derive this as a direct consequence of the uncertainty principle.
Antenna Beam Parameters

Pointing Accuracy
Δθ = rms pointing error

Primary beam $A(\theta)$

$\Delta \theta$

$\theta_{3\text{dB}}$
Beam Pattern

$l = \sin(\theta)$, $D = \text{antenna diameter in wavelengths}$

$\text{dB} = 10\log(\text{power ratio}) = 20\log(\text{voltage ratio})$

For VLA: $\theta_{3\text{dB}} = 1.02/D$, First null = 1.22/D

contours: $-3, -6, -10, -15, -20, -25, -30, -35, -40 \text{ dB}$