Chapter 16

Problem 91P. This is quite a hard problem. After all, what does the fact that the car resonates have to do with its rise when the weight is decreased? But the change in height due to a change in weight (mass) is related to the spring force. Thus, since $F = kx$ we know that

$$\Delta x = \frac{\Delta F}{k} = \frac{\Delta mg}{k}$$

which means we need the spring constant $k$ since we know the change in weight (720 pounds as given in the book’s problem). But the spring constant $k$ is involved in the natural frequency of the car’s springs ($k = \omega_0^2 M$). Here $M$ is the total mass of car plus occupants when the car is traveling down the road. We are told that the car is in resonance (Maximum Amplitude) so that we also know $\omega = \omega_0$. Therefore, our problem becomes only to find the angular frequency of the oscillations of the bouncing car. This is less difficult.

The period of oscillations is given by

$$T = \frac{d}{v} = \frac{13\text{ ft}}{10\text{ mph}}$$

We have a serious problem with units here and need to transform 10 MPH to feet per second. A convenient number to remember is that 60 MPH = 88 ft/sec. Therefore, dividing this by 6 we get 10 MPH = 14.67 ft/sec. If I put this in, I get $T = 0.886$ seconds for the period. But then the angular frequency is easily found from

$$\omega = \frac{2\pi}{T} = 7.09 \text{ rads/sec}$$

Thus

$$k = \omega_0^2 M = 50.27 \times \frac{2920}{32} = 4587 \text{ lb/ft}$$

Finally therefore,

$$\Delta x = \frac{\Delta mg}{k} = 0.157 \text{ ft} = 1.88 \text{ inches}$$

Problem 2. a) We are given the values of $k$ and of $m$ so we automatically have the values we need for the natural frequency. The numbers that WebAssign has given me are $k = 100$ N/m and $m = 0.15$ kg. Thus
\[ \omega_o = \sqrt{\frac{k}{m}} = 25.8 \text{ rad/s} \]

b) Again purely a definition. The damping constant is \( b = 2 \) in my problem:

\[ \Gamma = \frac{b}{m} = 13.3 \]

for my numbers.

c) The absorptive coefficient is defined as

\[ A_{ab} = \frac{F_0}{m} \frac{\Gamma \omega}{(\omega_o^2 - \omega^2)^2 + \Gamma^2 \omega^2} \]

and this is easy to calculate as we know every term. We already know \( \omega_o = 25.8 \) and \( \Gamma = 13.3 \). In my problem I am given \( F_0 = 2 \) and \( \omega = 20 \). Thus it is easy (in principle) to plug in the numbers and get the answer. But it has been my experience that trying to do all of this in one calculator operation is susceptible to serious error. Therefore I try to do my calculations for long formulas in several steps I writing down the numbers I calculate for each term making sure that they are reasonable numbers and then perform further algebraic operations as I go. Just to demonstrate here I will write out

\[ (\omega_o^2 - \omega^2)^2 = (666 - 400)^2 = 7.08 \times 10^4 \]

\[ \Gamma \omega = 266 \]

\[ \frac{F_0}{m} = 13.3 \]

I get

\[ A_{ab} = \frac{13.3 \times 266}{7.08 \times 10^4 + 7.08 \times 10^4} = .025 \]

This answer I trust.

d) Similarly

\[ A_{el} = \frac{F_0}{m} \frac{(\omega_o^2 - \omega^2)}{(\omega_o^2 - \omega^2)^2 + \Gamma^2 \omega^2} = .025 \]

I have fixed the numbers so that the two amplitudes are equal.

e) The Power is a simpler function, namely
\[ P = \frac{1}{2} F_\omega A_{ab} = 0.5 \text{ Watts} \]

f) The energy per cycle seems to have caused people some trouble but it should not be hard. Energy is just power times time. The time involved is one cycle. The angular frequency of oscillation is 20 rad/sec so the period is easily seen to be 0.314 seconds. Then the energy per cycle is just \( 0.5 \text{ Watts} \times 0.314 \text{ seconds} = 0.157 \text{ Joules} \). Not so hard when you see it.

**Problem 3.** Here we are asked to take the formula for Power\( \tilde{\Gamma} \) differentiate it and find the value of \( \omega \) for which the power is maximum. This is the power input from the driver and also of course the power dissipated in friction or whatever other dissipative forces might exist. The system is not a permanent storehouse of energy and therefore what comes in (on average) must be what goes out (on average).

Since

\[
P = \frac{1}{2} F_\omega A_{ab} = \frac{1}{2} F_\omega \frac{F_\omega}{m} \frac{\Gamma \omega}{(\omega_o^2 - \omega^2)^2 + \Gamma^2 \omega^2} = \frac{1}{2} \frac{F_\omega^2}{\Gamma m} \frac{\Gamma^2 \omega^2}{(\omega_o^2 - \omega^2)^2 + \Gamma^2 \omega^2}
\]

For the sake of symmetry I have written \( \Gamma^2 \omega^2 \) in the numerator and put an extra \( \Gamma \) in the denominator. Not so important but a nice aesthetic step.

When it comes to differentiating this I will ignore the constant term

\[
\frac{1}{2} \frac{F_\omega^2}{\Gamma m}
\]

and ignore the \( \Gamma^2 \) term in the numerator. Then I just have the differentiation of a fraction which I then set to zero. Because I set it to zero I can **ignore the denominator squared term** which always accompanies the derivative of a fraction! Thus I will have bottom times the derivative of the top - top times the derivative of the bottom set equal to zero!! I get

\[
\left[ (\omega_o^2 - \omega^2)^2 + \Gamma^2 \omega^2 \right] 2\omega - \omega^2 \left[ 2 \left( \omega_o^2 - \omega^2 \right) (-2\omega) + 2\Gamma^2 \omega \right] = 0
\]

The two terms must then be equal to each other and I get (cancelling some of the 2s that are around)

\[
\omega \left[ \omega_o^4 - 2\omega_o^2 \omega^2 + \omega^4 + \Gamma^2 \omega^2 \right] = \omega^2 \left[ (\omega_o^2 - \omega^2)(-2\omega) + \Gamma^2 \omega \right]
\]

The first thing I do is to cancel one power of \( \omega \) from each side. I have

\[
\left[ \omega_o^4 - 2\omega_o^2 \omega^2 + \omega^4 + \Gamma^2 \omega^2 \right] = \omega \left[ (\omega_o^2 - \omega^2)(-2\omega) + \Gamma^2 \omega \right] = \left[ (\omega_o^2 - \omega^2)(-2\omega) + \Gamma^2 \omega \right]
\]
The first thing I notice is a term which is $\Gamma^2 \omega^2$ on each side and therefore such terms cancel. I am left with

$$\omega_o^4 - 2\omega_o^2 \omega^2 + \omega^4 = -2\omega_o^2 \omega^2 + 2\omega^4$$

Once again there is a cancellation this time of the cross terms. I am finally left with $\omega_o^4 = \omega^4$. Since I cannot have $\omega < 0$ I am forced to say that the only solution is $\omega = \omega_o$, my desired answer.

**Problem 4.** The power curve represents a peak because I have just calculated the maximum which occurs at $\omega = \omega_o$. Then I can write that

$$P_{\text{max}} = \frac{1}{2} \frac{F_o^2 \Gamma^2 \omega^2}{m \Gamma^2 \omega^2} = \frac{1}{2} \frac{F_o^2}{m}$$

If I do this then I can write the power anywhere as

$$P = \frac{1}{2} \frac{F_o^2 \Gamma^2 \omega^2}{m (\omega_o^2 - \omega^2)^2 + \Gamma^2 \omega^2} = P_{\text{max}} \frac{\Gamma^2 \omega^2}{(\omega_o^2 - \omega^2)^2 + \Gamma^2 \omega^2}$$

so in fact there was little method to my madness when I put the $\Gamma$ in the denominator before not just aesthetics. To get the half maximum points then I set

$$\frac{1}{2} P_{\text{max}} = P_{\text{max}} \frac{\Gamma^2 \omega^2}{(\omega_o^2 - \omega^2)^2 + \Gamma^2 \omega^2}$$

which is very convenient. I can then cancel $P_{\text{max}}$ and cross multiply leaving me with

$$2\Gamma^2 \omega^2 = (\omega_o^2 - \omega^2)^2 + \Gamma^2 \omega^2$$

or

$$\Gamma^2 \omega^2 = (\omega_o^2 - \omega^2)^2$$

I can take the square root of both sides remembering that this gives us a $\pm$ sign! Then

$$(\omega_o^2 - \omega^2) = \pm \Gamma \omega$$

I can write this out as a quadratic equation in $\omega$ as

$$\omega^2 \mp \Gamma \omega - \omega_o^2 = 0$$

Use the quadratic formula and get

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\[ \omega = \pm \Gamma \pm \sqrt{\Gamma^2 + 4\omega_0^2} \]

Now I need a little logic. The term in the radical is \( \Gamma^2 + 4\omega_0^2 \) which is surely \( > \Gamma^2 \). Therefore if I choose the \(-\) sign in front of the radical (the square root) then I will surely have a negative value of \( \omega \) which is not allowed! So I use only the \(+\) sign here and get finally

\[ \omega = \sqrt{\omega_0^2 + \frac{\Gamma^2}{4}} \pm \frac{\Gamma}{2} \]

I have just put the 2 from the denominator into the equation - again for aesthetic purposes. My two value of \( \omega_{1/2} \) are then

\[ \sqrt{\omega_0^2 + \frac{\Gamma^2}{4}} + \frac{\Gamma}{2} \]

and

\[ \sqrt{\omega_0^2 + \frac{\Gamma^2}{4}} - \frac{\Gamma}{2} \]

If I subtract these two value I get the full width half maximum (half height) of the resonance curve!

This value if

\[ (\Delta \omega)_{fwhm} = \Gamma \]

This is our final and quite profound result. We will meet it again in Quantum Mechanics.